

The number of eigenstates: counting function and heat kernel

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The number of eigenstates: counting function and heat kernel

Wu-Sheng Dai^{ab} and Mi Xie^{ab}

^a*Department of Physics, Tianjin University,
Tianjin 300072, P.R. China*

^b*LiuHui Center for Applied Mathematics, Nankai University & Tianjin University,
Tianjin 300072, P.R. China*

E-mail: daiwusheng@tju.edu.cn, xiemi@tju.edu.cn

ABSTRACT: The main aim of this paper is twofold: (1) revealing a relation between the counting function $N(\lambda)$ (the number of the eigenstates with eigenvalue smaller than a given number) and the heat kernel $K(t)$, which is still an open problem in mathematics, and (2) introducing an approach for the calculation of $N(\lambda)$, for there is no effective method for calculating $N(\lambda)$ beyond leading order. We suggest a new expression of $N(\lambda)$ which is more suitable for practical calculations. A renormalization procedure is constructed for removing the divergences which appear when obtaining $N(\lambda)$ from a nonuniformly convergent expansion of $K(t)$. We calculate $N(\lambda)$ for D -dimensional boxes, three-dimensional balls, and two-dimensional multiply-connected irregular regions. By the Gauss-Bonnet theorem, we generalize the simply-connected heat kernel to the multiply-connected case; this result proves Kac's conjecture on the two-dimensional multiply-connected heat kernel. The approaches for calculating eigenvalue spectra and state densities from $N(\lambda)$ are introduced.

KEYWORDS: Differential and Algebraic Geometry, Boundary Quantum Field Theory.

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1. Introduction

A problem stemming from physics and soon becoming an important mathematical problem is to recover geometry of a manifold from the knowledge of the eigenvalues of a natural differential operator. This problem originates in the theory of radiation (how to determine the state density of the electromagnetic wave in a given cavity) and is formulated by Kac as "Can one hear the shape of a drum?" [1].

What the original problem asked is that how many eigenvalues are smaller than a given number λ . This problem is formulated as to seek the so-called counting function $N(\lambda)$. For a given spectrum $\{\lambda_n\}$, the counting function is directly defined to be

$$N(\lambda) = \text{the number of eigenvalues (with multiplicity) smaller than } \lambda, \quad (1.1)$$

i.e.,

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1. \quad (1.2)$$

In other words, the counting function represents the number of the eigenstates whose eigenvalues are smaller than λ . Nevertheless, the counting function $N(\lambda)$ is very difficult to calculate and there is no general method for calculating $N(\lambda)$ in mathematics [2]. This is because when calculating $N(\lambda)$, one often encounters some unsolved problems in number theory. For example, when calculating the counting function for the spectrum of the Laplace operator on a tori, one encounters the Gauss circle problem in number theory. The Hardy-Littlewood-Karamata Tauberian theorem gives the first term of the asymptotic expansion of $N(\lambda)$, but does not provide any information beyond the first-order term [2].

For a given spectrum $\{\lambda_n\}$, one also introduces the heat kernel,

$$K(t) = \sum_n e^{-\lambda_n t}, \quad (1.3)$$

which is another important function describing the relation between the eigenvalues of the operator and the geometrical property of the manifold. The heat kernel is relatively easy to calculate and some methods for calculating heat kernels have been developed [3]. In special, Elizalde *et al.* developed a very effective approach for calculating $K(t)$ [4–8].

There is no doubt that there must exist a relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$, since they are both determined by the same spectrum $\{\lambda_n\}$. Nevertheless, the relation between $N(\lambda)$ and $K(t)$ is still an open problem in mathematics [2].

Though the definition of counting function $N(\lambda)$, eq. (1.1) or (1.2), is comprehensive and intuitive, other than the definition of heat kernel $K(t)$, eq. (1.3), the definition of $N(\lambda)$ is not suitable for practical calculations. In fact, if starting from this definition to calculate the counting function, one may always encounter the mathematical difficulty mentioned above. In this paper, we will propose an expression for $N(\lambda)$, which has a similar form with the definition formula of $K(t)$, eq. (1.3), and is more operable than the definition (1.2). This expression can be regarded as an alternative definition formula for $N(\lambda)$.

As a main result of this paper, we provide a relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$. This problem interests us from both purely mathematical viewpoint and practical viewpoint. This relation allows us to calculate $N(\lambda)$ from $K(t)$ which, as mentioned above, is often relatively easier to calculate.

The relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$ presented in this paper is an integral transformation. However, in practice the heat kernel is often given in the form of a series expansion and in most cases only the first several expansion coefficients can be obtained [3]. This requires us to integrate term by term. Nevertheless, the series expansion of $K(t)$ is not convergent uniformly. As a result, the integral of some terms will be divergent. The counting function $N(\lambda)$ is of course finite; the divergences are caused by illegally integrating a nonuniformly convergent series term by term. For dealing with this problem, we provide a renormalization procedure to remove the divergences.

In fact, the results of this paper provide two approaches for calculating the counting function $N(\lambda)$: (1) by the expression for $N(\lambda)$ presented in this paper, one can calculate $N(\lambda)$ directly, and (2) by the relation between $N(\lambda)$ and $K(t)$, one can first calculate the heat kernel $K(t)$ and then calculate $N(\lambda)$ from $K(t)$. As a comparison, we calculate $N(\lambda)$ for a D -dimensional box by both these two approaches, respectively.

As an example of the calculation of counting functions by the relation between $N(\lambda)$ and $K(t)$, we calculate $N(\lambda)$ from $K(t)$ for three-dimensional balls.

As another example, we calculate the counting function from the heat kernel for the minus Laplace operator in a two-dimensional region with irregular shape and nontrivial topology. For two-dimensional heat kernels, there are two known results: the heat kernel for a simply-connected region [9] and a hypothesis made by Kac on the heat kernel for the multiply-connected case [1]. In the present paper, we first generalize the heat kernel for a simply-connected region given in [9] to the case of a multiply-connected region which is bounded by a smooth but irregular curve and with some holes. This result proves Kac's conjecture. Then, we calculate the counting function for the two-dimensional multiply-connected region from the heat kernel by the relation between $N(\lambda)$ and $K(t)$.

As applications, we discuss the problem that how to calculate the asymptotic expressions for eigenvalue spectra and state densities from the counting function $N(\lambda)$. By $N(\lambda)$, we construct an equation for eigenvalues and then provide an approximate solution in which the eigenvalue is expressed as a function of heat kernel coefficients. Moreover, we also provide two expressions for state densities by the relation between $N(\lambda)$ and $K(t)$ and by the expression of $N(\lambda)$ provided in the present paper.

The counting function and the heat kernel are interesting in both mathematics and physics:

In mathematics, the relation between the spectrum of the Laplace operator on a Riemannian manifold and the geometry of this Riemannian manifold is an important subject [2, 10–12], and the problem of spectral asymptotics is one of the central problems in the theory of partial differential operators [13]. One of the main problems is to seek the asymptotic expansions of the counting function $N(\lambda)$ and the heat kernel $K(t)$. The general relation between $N(\lambda)$ and $K(t)$ is still unknown. Especially, there is no general method for calculating the asymptotic expansion of the counting function $N(\lambda)$.

In physics, the spectrum of the Laplace operator on a Riemannian manifold can be directly applied to boundary quantum fields. Moreover, reconstructing the geometrical property of a system from an eigenproblem is an interesting and important problem. For example, Aurich *et al.* reconstruct the shape of the universe from the result of the eigenproblem [14]. The counting function is directly related to the Casimir effect, the spectrum problem, and the state density, etc. Moreover, there are many studies on heat kernels. In quantum field theory, it is of crucial importance to evaluate the one-loop divergences, and the control of the ultra-violet divergences can be achieved by using heat kernel regularization methods [3, 15, 16]. The heat kernel expansion becomes a standard tool in the calculations of vacuum energies [17], the Casimir effect [18, 19], and quantum anomalies [20]. Now a large amount of research has been devoted to quantum gravity based on the heat kernel expansion, including semiclassical approaches [21], black hole thermody-

namics [22], wormhole physics [23], heat kernels in curved space [24], and supergravity theories [25, 26]. In addition, the heat kernel expansion is also used to study string theory [27–29] and noncommutative field theory [30–32]. The relation between the counting function and the heat kernel allows one to introduce the concept of the counting function into these fields. Moreover, there exist some indirect ways to construct the heat kernel without solving the heat equation, and the heat kernel coefficients for various manifolds have been discussed [3, 6, 33, 34], which makes the heat kernel expansion a very convenient and effective tool. By the relation provided in the present paper, such results of the heat kernel can be directly applied to the problem of the counting function.

In section 2, we provide an expression for the counting function $N(\lambda)$, which is more operable compared to the definition (1.2) and can be regarded as an alternative definition formula for the counting function. In section 3, we point out a relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$. In section 4, we provide a renormalization procedure for removing the divergences appearing in the calculation of $N(\lambda)$ from a nonuniformly convergent series expansion of $K(t)$. In section 5, we compare the two approaches for calculating $N(\lambda)$ using a D -dimensional box as an example. In section 6, we calculate the counting function of a three-dimensional ball from the heat kernel. In section 7, for two-dimensional cases, we first generalize the heat kernel in a simply-connected region to a multiply-connected region, and then calculate the counting function in a multiply-connected region from the heat kernel. The result of the heat kernel in the multiply-connected region obtained in the present paper proves Kac’s hypothesis. In sections 8 and 9, as applications of the counting function, we discuss how to calculate the asymptotic expressions for spectra and state densities from $N(\lambda)$. The conclusions are summarized in section 10.

2. An expression for the counting function: an alternative definition formula

The definition formula for the counting function $N(\lambda)$, eq. (1.2), is not suitable for practical calculations. In fact, starting from this definition to calculate $N(\lambda)$, one may encounter many difficulties, e.g., the unsolved problems in number theory [2]. Contrarily, the definition formula for the heat kernel, eq. (1.3), is more operable. In this section, we present an expression for $N(\lambda)$, which has a similar form with that of the heat kernel $K(t)$, eq. (1.3). Such an expression can be regarded as an operable definition formula for $N(\lambda)$.

Theorem 1.

$$N(\lambda) = \lim_{\beta \rightarrow \infty} \sum_n \frac{1}{e^{\beta(\lambda_n - \lambda)} + 1}. \tag{2.1}$$

Proof. Observing that

$$\lim_{\beta \rightarrow \infty} \frac{1}{e^{\beta(\lambda_n - \lambda)} + 1} = \begin{cases} 1, & \text{when } \lambda_n < \lambda, \\ 0, & \text{when } \lambda_n > \lambda, \end{cases}$$

we have

$$\lim_{\beta \rightarrow \infty} \sum_n \frac{1}{e^{\beta(\lambda_n - \lambda)} + 1} = \sum_{\lambda_n < \lambda} 1 = N(\lambda).$$

Comparing with eq. (1.2), eq. (2.1) is obviously more operable, since eq. (2.1) converts the partial sum in eq. (1.2), $\sum_{\lambda_n < \lambda}$, into a sum over all possible values, $\sum_{\lambda_n < \infty}$. This will, of course, make the calculation easier.

3. The relation between counting function and heat kernel

In this section, we provide a relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$. This relation is an interesting mathematical result and is useful for practical calculations. As mentioned above, the calculation of the counting function $N(\lambda)$ is more difficult than that of the heat kernel $K(t)$. With the relation between $N(\lambda)$ and $K(t)$ given in the following, one can first calculate $K(t)$ and then calculate $N(\lambda)$ from the result of $K(t)$.

The relation between $N(\lambda)$ and $K(t)$ is as follows.

Theorem 2.

$$K(t) = t \int_0^\infty N(\lambda) e^{-\lambda t} d\lambda. \tag{3.1}$$

Proof. The generalized Abel partial summation formula reads

$$\sum_{u_1 < \lambda_n \leq u_2} b(n) f(\lambda_n) = B(u_2) f(u_2) - B(u_1) f(u_1) - \int_{u_1}^{u_2} B(u) f'(u) du, \tag{3.2}$$

where $\lambda_i \in \mathbb{R}$, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$, and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. $f(u)$ is a continuously differentiable function on $[u_1, u_2]$ ($0 \leq u_1 < u_2$, $\lambda_1 \leq u_2$), $b(n)$ ($n = 1, 2, 3, \dots$) are arbitrary complex numbers, and $B(u) = \sum_{\lambda_n \leq u} b(n)$. We apply the generalized Abel partial summation formula, eq. (3.2), with $f(u) = e^{-u(s-s_0)}$ and $b(n) = a_n e^{-\lambda_n s_0}$, where $s, s_0 \in \mathbb{C}$. Then

$$\begin{aligned} A(u_2, s) - A(u_1, s) &= A(u_2, s_0) e^{-u_2(s-s_0)} - A(u_1, s_0) e^{-u_1(s-s_0)} \\ &\quad + (s - s_0) \int_{u_1}^{u_2} A(u, s_0) e^{-u(s-s_0)} du, \end{aligned} \tag{3.3}$$

where

$$A(u, s) = \sum_{\lambda_n \leq u} a_n e^{-\lambda_n s}. \tag{3.4}$$

Setting $a_n = 1$ in eq. (3.4), we find

$$A(\lambda, 0) = \sum_{\lambda_n \leq \lambda} 1 = N(\lambda),$$

the counting function, and

$$A(\infty, t) = \sum_n e^{-\lambda_n t} = K(t),$$

the heat kernel. By eq. (3.4), we also have $A(0, t) = 0$. Then, by eq. (3.3), we have

$$K(t) = A(\infty, t) - A(0, t) = t \int_0^\infty N(\lambda) e^{-\lambda t} d\lambda. \quad (3.5)$$

This is just eq. (3.1).

Theorem 3.

$$N(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(t) \frac{e^{\lambda t}}{t} dt, \quad c > \lim_{n \rightarrow \infty} \frac{\ln n}{\lambda_n}. \quad (3.6)$$

Proof. By the Perron formula, we have

$$\sum_{\mu_n < x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(t) \frac{x^t}{t} dt, \quad (3.7)$$

where

$$f(s) = \sum_{n=1}^\infty \frac{a_n}{\mu_n^s}, \quad (3.8)$$

and c is a constant which is greater than the abscissa of absolute convergence of the Dirichlet series $f(s)$. Setting

$$a_n = 1 \text{ and } \mu_n = e^{\lambda_n}$$

in eq. (3.8), we obtain the heat kernel,

$$f(t) = \sum_{n=1}^\infty e^{-\lambda_n t} = K(t).$$

The abscissa of absolute convergence of $f(t)$ equals its abscissa of convergence, equaling $\overline{\lim}_{n \rightarrow \infty} \ln n / \lambda_n = \lim_{n \rightarrow \infty} \ln n / \lambda_n$. Thus, by eq. (3.7), we have

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(t) \frac{e^{\lambda t}}{t} dt,$$

and $c > \lim_{n \rightarrow \infty} \ln n / \lambda_n$. This proves the theorem.

Clearly, the relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$ obtained above is an integral transformation and the corresponding inverse transformation.

4. Calculating $N(\lambda)$ from an asymptotics of $K(t)$

In principle, we can calculate the counting function $N(\lambda)$ from the heat kernel $K(t)$ or calculate $K(t)$ from $N(\lambda)$ by the relation (3.6) or (3.1) directly. However, the relation given above is an integral transformation. Therefore, when the heat kernel $K(t)$ is in the form of a series expansion, practically, the above relation is available only when this integral transformation can be applied to each term of the series, or, in other words, this series must can be integrated term by term. If the series is not uniformly convergent, even though the integral of the sum function is convergent, the integrals of some of the terms may be divergent. In this case, we need a renormalization procedure to remove the divergences.

4.1 An asymptotics for $N(\lambda)$

Concretely, the expansion of heat kernel $K(t)$ can be expressed as [6]

$$K(t) = (4\pi t)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} B_k t^k, \tag{4.1}$$

where B_k is the heat kernel coefficient, D is the dimension of space. In fact, in practice only the first several heat kernel coefficients can be obtained, so we have to integrate term by term. Substituting eq. (4.1) into eq. (3.6) and exchanging the order of integration and summation gives

$$N(\lambda) = (4\pi)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} B_k I_D(k), \tag{4.2}$$

where

$$I_D(k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-D/2+k} \frac{e^{\lambda t}}{t} dt. \tag{4.3}$$

Nevertheless, for a power series, unless the series is uniformly convergent, it is not permissible to integrate term by term, i.e., the order of integration and summation cannot be exchanged. When calculating an integral of a series that is not uniformly convergent term by term, one may find that the integrals of some terms diverge. In our case, it can be directly seen that only when $k < D/2 + 1$, $I_D(k)$ is convergent. That is to say, the term with $k \geq D/2 + 1$ needs to be renormalized for obtaining a finite result.

4.2 Renormalization

The integral (4.3) is divergent when $k \geq D/2 + 1$. For obtaining the counting function $N(\lambda)$ from the series (4.2), we need to make sense of these divergent integrals. In other words, we need a renormalization procedure to remove the divergences.

It can be directly seen that eq. (4.3) is the inverse Laplace transformation of the function $t^{-D/2+k-1}$. Thus for $k < D/2 + 1$, we have [35]

$$I_D(k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-D/2+k-1} e^{\lambda t} dt = \frac{1}{\Gamma(1 + D/2 - k)} \lambda^{D/2-k}, \quad (k < D/2 + 1). \tag{4.4}$$

The definition domain for the gamma function $\Gamma(1 + D/2 - k)$ is $k < D/2 + 1$. For $k \geq D/2 + 1$, we can redefine the gamma function by analytically continuing the gamma function via the recurrence relation $\Gamma(x) = \frac{1}{x} \Gamma(x + 1)$ [36]. Concretely, when defining the gamma function as $\Gamma(x) = \frac{1}{x} \Gamma(x + 1)$, the definition domain of the gamma function is changed to $k < D/2 + 2$. Repeating this procedure will eventually analytically continue the gamma function to the whole real axis.

From the above result we can see that, like that in the perturbative expansion of the scattering amplitude in quantum field theory, the first several terms are convergent (like tree diagrams) and the other terms are divergent and needed to be renormalized (like loop diagrams). For clarity, write the counting function $N(\lambda)$ in two parts:

$$N(\lambda) = N_T(\lambda) + N_L(\lambda), \tag{4.5}$$

where

$$N_T(\lambda) = (4\pi)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\frac{D}{2} + \frac{1}{2}} B_k I_D(k) \quad (4.6)$$

is the convergent part and

$$N_L(\lambda) = (4\pi)^{-D/2} \sum_{k=\frac{D}{2}+1, \frac{D}{2}+\frac{3}{2}, \dots}^{\infty} B_k I_D^R(k) \quad (4.7)$$

is the renormalized divergent part, where $I_D^R(k)$ denotes the renormalized result. $N_L(\lambda)$ is only a higher-order contribution to the counting function and is often negligible.

After renormalization, eqs. (4.6) and (4.7) can be written as

$$N(\lambda) = \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} C_k \lambda^{D/2-k} \quad (4.8)$$

with the counting function coefficient

$$C_k = (4\pi)^{-D/2} \frac{B_k}{\Gamma(1 + D/2 - k)}. \quad (4.9)$$

5. Comparison of the two approaches for calculating $N(\lambda)$: D -dimensional boxes

The results presented in sections 2 and 3 show that there are two approaches for calculating the counting function $N(\lambda)$. The first is to directly calculate $N(\lambda)$ by eq. (2.1), the operable expression of $N(\lambda)$ presented in this paper; the second is to first calculate the heat kernel $K(t)$ and then to calculate $N(\lambda)$ from $K(t)$ by the relation between $N(\lambda)$ and $K(t)$. In this section, as an example, we calculate $N(\lambda)$ by the above two approaches respectively for a rectangular D -dimensional box.

The spectrum of the minus Laplace operator of a D -dimensional rectangle box reads

$$\lambda_{n_1, n_2, \dots, n_D} = \pi^2 \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \dots + \frac{n_D^2}{L_D^2} \right), \quad n_i = 1, 2, \dots, \quad (5.1)$$

where L_i is the side length.

First, we directly calculate the counting function $N(\lambda)$ from eq. (2.1):

$$N(\lambda) = \lim_{\beta \rightarrow \infty} \sum_{n_1, n_2, \dots, n_D} \frac{1}{e^{\beta(\lambda_{n_1, n_2, \dots, n_D} - \lambda)} + 1}. \quad (5.2)$$

By the Euler-MacLaurin formula $\sum_{n=0}^{\infty} F(n) = \int_0^{\infty} F(n) dn + \frac{1}{2}F(0) - \frac{1}{2!}B_2F'(0) - \frac{1}{4!}B_4F'''(0) + \dots$, where B_2, B_4, \dots are Bernoulli numbers, we achieve

$$N(\lambda) = (4\pi)^{-D/2} \left[\frac{V}{\Gamma(D/2 + 1)} \lambda^{D/2} - \sum_{i=1}^D \frac{V}{L_i} \frac{\sqrt{\pi}}{\Gamma(D/2 + 1/2)} \lambda^{D/2-1/2} + \sum_{i < j}^D \frac{V}{L_i L_j} \frac{\pi}{\Gamma(D/2)} \lambda^{D/2-1} + \dots + (-1)^D \pi^{D/2} \right], \quad (5.3)$$

where $V = L_1 L_2 \cdots L_D$ is the volume.

Alternatively, we can first calculate the heat kernel from the definition formula of $K(t)$,

$$K(t) = \sum_{n_1, n_2, \dots, n_D} e^{-\lambda(n_1, n_2, \dots, n_D)t}. \tag{5.4}$$

A direct calculation by the Euler-MacLaurin formula gives the heat kernel coefficients:

$$\begin{aligned} B_0 &= V, \\ B_{1/2} &= -\sqrt{\pi} \sum_{i=1}^D \frac{V}{L_i}, \\ B_1 &= \pi \sum_{i < j}^D \frac{V}{L_i L_j}, \dots, \\ B_{\nu/2} &= (-1)^\nu \pi^{\nu/2} \sum_{i_1 < i_2 < \dots < i_\nu}^D \frac{V}{L_{i_1} L_{i_2} \cdots L_{i_\nu}}. \end{aligned} \tag{5.5}$$

By the relation between the counting function coefficient and the heat kernel coefficient, eq. (4.9), we also achieve eq. (5.3).

From the above result, we can see the equivalence between these two approaches.

6. Calculating $N(\lambda)$ from $K(t)$: three-dimensional balls

In this section, as an example, we calculate the counting function $N(\lambda)$ from the series expansion of the heat kernel $K(t)$ for a three-dimensional ball. It should be emphasized that for calculating the heat kernel $K(t)$, it is not needed to know the eigenvalue spectrum in advance [6].

For a three-dimensional ball, from eq. (4.9), the expansion coefficients of the counting function $N(\lambda)$ can be obtained as

$$C_k = (4\pi)^{-3/2} \frac{B_k}{\Gamma(5/2 - k)}. \tag{6.1}$$

The heat kernel coefficients for Dirichlet, Neumann, and Robin boundary conditions for three-dimensional balls are calculated explicitly in [6].

Take the case of Dirichlet boundary condition as an example. Ref. [6] gives the first 21 heat kernel coefficients. From this, we can obtain the first 21 counting function coefficients C_k by eq. (6.1). Only taking the tree-diagram-like part into account, from eq. (4.6), we have

$$N(\lambda) = \frac{2}{9\pi} R^3 \lambda^{3/2} - \frac{1}{4} R^2 \lambda + \frac{2}{3\pi} R \lambda^{1/2} - \frac{1}{48} - \frac{2}{315\pi} \frac{1}{R \lambda^{1/2}}. \tag{6.2}$$

It can be directly checked that the tree-diagram-like part provides the main contribution to the $N(\lambda)$.

7. Multiply-connected cases: two-dimensional $K(t)$ and $N(\lambda)$

An interesting special case is about two-dimensional counting function and two-dimensional heat kernel, which is just the original problem formulated by Kac as "Can one hear the shape of a drum?". The first result of this problem is proved by Weyl [37] and then improved by Pleijel [38] and Kac [1]. In [1], Kac pointed out a special relation between $N(\lambda)$ and $K(t)$: in two dimensions, the leading-order term of $N(\lambda)$ is accidentally equal to the leading-order term of $K(t)$. However, beyond the leading order, there is no further result. Moreover, for the multiply-connected case, Kac made a hypothesis about the topological contribution on the heat kernel $K(t)$: the contribution from the nontrivial topology (connectivity) is in proportion to the Euler-Poincaré characteristic number.

In this section, (1) with the help of the relation between $N(\lambda)$ and $K(t)$ given above, we give a proof for Kac's result: the leading-order term of $N(\lambda)$ equals the leading-order term of $K(t)$; (2) based on the Gauss-Bonnet theorem, we first generalize the simply-connected heat kernel $K(t)$ given in [9] to the multiply-connected case (this result proves Kac's conjecture: in two dimensions, the topology contribution is proportion to the Euler-Poincaré characteristic number), and, then, calculate the two-dimensional $N(\lambda)$ in a multiply-connected region with the help of the relation between $N(\lambda)$ and $K(t)$.

7.1 The leading contribution of $N(\lambda)$ and $K(t)$ in two dimensions

Using the relation between $N(\lambda)$ and $K(t)$, we first prove that when the number of eigenstates per unit interval (the density of eigenstates) $\rho(\lambda)$ is a constant, $N(\lambda)$ equals $K(t)$ in the limit $\lambda \rightarrow \infty$ or $t \rightarrow 0$, i.e., in the limit $\lambda \rightarrow \infty$ or $t \rightarrow 0$,

$$N(\lambda) = K\left(\frac{1}{\lambda}\right) \text{ or } N\left(\frac{1}{t}\right) = K(t). \tag{7.1}$$

The proof is straightforward. In the limit $\lambda \rightarrow \infty$ or $t \rightarrow 0$, the summations can be converted into integrals:

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1 = \int_0^\lambda \rho(\lambda') d\lambda', \tag{7.2}$$

$$K(t) = \sum_n e^{-\lambda_n t} = \int_0^\infty \rho(\lambda') e^{-\lambda' t} d\lambda'. \tag{7.3}$$

If $\rho(\lambda) = C$, where C is a constant, then

$$N(\lambda) = C\lambda, \tag{7.4}$$

$$K(t) = \frac{C}{t}. \tag{7.5}$$

This proves eq. (7.1).

In two dimensions, the state density for the eigenstate of the minus Laplace operator is a constant, so the leading contribution of the counting function equals the leading contribution of the heat kernel. This is just the case appeared in Kac's work [1].

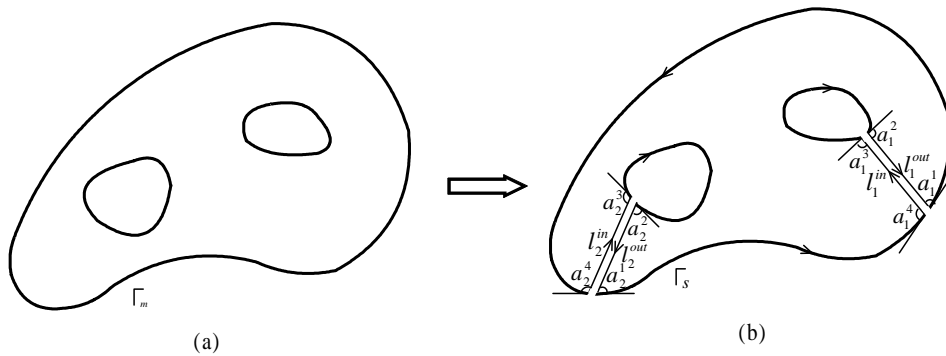


Figure 1: Converting a multiply-connected region to a simply-connected region.

7.2 $K(t)$ and $N(\lambda)$ in Multiply-connected regions

For achieving the counting function in a multiply-connected region by the relation between $K(t)$ and $N(\lambda)$, we need to start with the heat kernel in a multiply-connected region. For two-dimensional heat kernels, there are two known results: in the simply-connected case, ref. [9] provides a series expansion of the heat kernel; in the multiply-connected case, Kac makes a hypothesis on the heat kernel [1]. In the following, based on the Gauss-Bonnet theorem, we first generalize the simply-connected result of heat kernel $K(t)$ given in [9] to the multiply-connected case. The result can be viewed as a proof of Kac’s hypothesis.

In the simply-connected case, the heat kernel for a two-dimensional plane bounded by a smooth curve Γ_s and with the Dirichlet boundary condition on Γ_s reads [9]

$$K(t) = \frac{S}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{12\pi} \int_{\Gamma_s} k(s) ds + \dots, \tag{7.6}$$

where S is the area of the region, L is the length of Γ_s , $k(s)$ is the curvature of the curve Γ_s at the point s .

To generalize the result of the simply-connected region, eq. (7.6), to the multiply-connected case, we first convert the multiply-connected region bounded by Γ_m (figure 1a) to a simply-connected one bounded by a piecewise smooth simple closed curve Γ_s (figure 1b). In the simply-connected region illustrated in figure 1b, the first two terms which are proportional to the area and the perimeter of the region, respectively, can be calculated by the method in the simply-connected case directly. For the third term, by the Gauss-Bonnet theorem, we can calculate the integral of the curvature along Γ_s . The Gauss-Bonnet theorem reads [39]

$$\sum_i (\pi - a_i) + \int_{\Gamma_s} k(s) ds + \iint K d\sigma = 2\pi\chi, \tag{7.7}$$

where a_i is the interior angle of Γ_s at each vertex, $K = 0$ in the present case is the Gauss curvature, and χ is the Euler-Poincaré characteristic number. From figure 1, since the integrals along l_i^{in} and l_i^{out} ($i = 1, \dots, r$) cancel each other in pairs, we achieve

$$\int_{\Gamma_m} k(s) ds = \int_{\Gamma_s} k(s) ds, \tag{7.8}$$

where r is the number of holes in the region. Moreover, it is also easy to see that

$$\sum_{i=1}^r \sum_{\alpha=1}^4 (\pi - a_i^\alpha) = r2\pi. \quad (7.9)$$

The region bounded by Γ_s in figure 1b is simply-connected, so $\chi = 1$. Thus, from eq. (7.7), we have

$$\int_{\Gamma_s} k(s) ds = 2\pi(1-r). \quad (7.10)$$

By the approach given by [20], we can calculate the first two terms. Then, with eqs. (7.6), (7.8), and (7.10), we have

$$K(t) = \frac{S}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1-r}{6}. \quad (7.11)$$

This is just the result that Kac hypothesized in [1].

By the relation between $N(\lambda)$ and $K(t)$, we can directly calculate the counting function in the two-dimensional multiply-connected region:

$$N(\lambda) = \frac{S}{4\pi} \lambda - \frac{L}{\pi} \sqrt{\lambda} + \frac{1-r}{6}. \quad (7.12)$$

8. Calculating spectrum from $N(\lambda)$

From the counting function $N(\lambda)$, we can directly obtain the asymptotic expression for the spectrum of the system [40]. In view of the fact that one can obtain the heat kernel $K(t)$ and, accordingly, obtain the counting function $N(\lambda)$ without knowing the spectrum in advance [6], this result can serve as an approach for calculating the spectrum.

The counting function is the number of eigenvalues smaller than λ , so for the n -th eigenvalue λ_n , we have

$$N(\lambda_n) = n. \quad (8.1)$$

Therefore, from eqs. (8.1) and (4.8), we have

$$(4\pi)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} \frac{B_k}{\Gamma(1 + D/2 - k)} \lambda_n^{D/2 - k} = n. \quad (8.2)$$

The eigenvalue λ_n can be solved from eq. (8.2).

For approximately solving eq. (8.2), we assume that λ_n can be expanded as

$$\lambda_n = \sum_{m=0}^{\infty} \alpha_m n^{(2-m)/D}. \quad (8.3)$$

Then we have

$$\begin{aligned} \lambda_n = & 4\pi\Gamma^{2/D} \left(\frac{D}{2} + 1 \right) \frac{1}{B_0^{2/D}} n^{2/D} - 4\sqrt{\pi} \frac{\Gamma^{1+1/D} \left(\frac{D}{2} + 1 \right)}{D\Gamma \left(\frac{D}{2} + \frac{1}{2} \right)} \frac{B_{1/2}}{B_0^{1+1/D}} n^{1/D} \\ & + \left[\frac{D\Gamma^2 \left(\frac{D}{2} \right)}{4\Gamma^2 \left(\frac{D}{2} + \frac{1}{2} \right)} \frac{B_{1/2}^2}{B_0^2} - \frac{B_1}{B_0} \right] n^0 + \dots \end{aligned} \quad (8.4)$$

As examples, we list some spectra obtained by this approach for various dimensional balls with Dirichlet, Neumann, and Robin boundary conditions, respectively. The heat kernel coefficients are given in [6].

For a three-dimensional ball, the spectrum for the Dirichlet and the Neumann or Robin boundary conditions is

$$\lambda_n = \frac{3}{2} (6\pi^2)^{1/3} \frac{1}{R^2} n^{2/3} \pm \frac{3}{8} (6\pi^2)^{2/3} \frac{1}{R^2} n^{1/3} + \left(\frac{27}{64} \pi^2 - 2 \right) \frac{1}{R^2}, \quad (8.5)$$

where R is the radius. In this equation and following, the upper sign stands for the Dirichlet boundary condition and the lower sign for the Neumann or Robin boundary condition.

For a four-dimensional ball, the spectrum is

$$\lambda_n = \frac{8}{R^2} n^{1/2} \pm \frac{16\sqrt{2}}{3} \frac{1}{R^2} n^{1/4} - \frac{26}{9} \frac{1}{R^2}, \quad (8.6)$$

and for a five-dimensional ball, the spectrum is

$$\lambda_n = \frac{1}{2} \left(450\sqrt{2}\pi \right)^{2/5} \frac{1}{R^2} n^{2/5} \pm \frac{15\pi}{32} (3600\pi)^{1/5} \frac{1}{R^2} n^{1/5} + \left(\frac{1125}{1024} \pi^2 - \frac{20}{3} \right) \frac{1}{R^2}. \quad (8.7)$$

9. The state density

In this section, we provide two approaches to achieve the state density based on the above results.

9.1 Calculating the state density from $N(\lambda)$ and $K(t)$

The first approach is straightforward. The meaning of the counting function $N(\lambda)$ is the number of the states whose eigenvalues are smaller than λ , so the state density reads

$$\rho(\lambda) = \frac{dN(\lambda)}{d\lambda}. \quad (9.1)$$

Thus, from eq. (4.8), we have

$$\rho(\lambda) = (4\pi)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} \frac{B_k}{\Gamma(D/2 - k)} \lambda^{D/2 - k - 1}. \quad (9.2)$$

From this result, one can obtain the state density once he knows the heat kernel coefficients.

9.2 Calculating the state density from the spectrum

An alternative way for obtaining the state density is to start with the expression of the counting function given in section 2.

Derivating both sides of eq. (2.1) by λ gives

$$\rho(\lambda) = \frac{dN(\lambda)}{d\lambda} = \lim_{\beta \rightarrow \infty} \sum_n \frac{\beta e^{\beta(\lambda_n - \lambda)}}{[e^{\beta(\lambda_n - \lambda)} + 1]^2}. \quad (9.3)$$

Or, approximately, we can convert the summation over n to an integral:

$$\rho(\lambda) = \lim_{\beta \rightarrow \infty} \int dn \frac{\beta e^{\beta(\lambda_n - \lambda)}}{[e^{\beta(\lambda_n - \lambda)} + 1]^2}. \quad (9.4)$$

Then, we can directly calculate the state density from the spectrum.

10. Conclusions

In this paper, we reveal a relation between the counting function $N(\lambda)$ and the heat kernel $K(t)$ and provide an operable expression for $N(\lambda)$. By the relation, one can calculate the counting function from a known heat kernel, and *vice versa*. By the expression of the counting function presented in this paper, one can achieve the counting function by a direct calculation.

The relation between $N(\lambda)$ and $K(t)$ is an integral transformation and its inverse transformation. When calculating $N(\lambda)$ from $K(t)$ by this relation, however, one may encounter the problem of divergence. This is because in most cases the expression of $K(t)$ is in the form of a power series (in fact, often only first several heat kernel coefficients can be obtained), but the series is not uniformly convergent. As a result, when applying the integral transformation to the series, one needs to integrate term by term. However, to integrate term by term is not feasible: the integration of some terms will diverge. For removing the divergences, we develop a renormalization procedure.

The results of this paper provide two approaches for calculating the counting functions: one is to calculate $N(\lambda)$ directly from the expression of $N(\lambda)$ given in section 2 and the other is based on the relation between $N(\lambda)$ and $K(t)$ given in section 3. As a comparison between the two approaches, we calculate the counting function for D -dimensional boxes by these two approaches, respectively.

As applications of the relation between $N(\lambda)$ and $K(t)$, we also calculate the counting functions for three-dimensional balls and two-dimensional multiply-connected irregular regions. For calculating the counting function of a two-dimensional multiply-connected region, we generalize the result of the heat kernel of a two-dimensional simply-connected region to a multiply-connected one, and the result proves Kac's conjecture on the two-dimensional multiply-connected heat kernel.

Moreover, we also present approaches for calculating eigenvalue spectra and state densities from the counting function.

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Erratum

Eq. (4.8) should be

$$N(\lambda) = \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} C_k \lambda^{D/2-k} + \sum_{l=0, 1, 2, \dots}^{\infty} (4\pi)^{-D/2} B_{1+D/2+l} \delta^{(l)}(\lambda),$$

because when $-D/2 + k - 1$ is a non-negative integer, the integral (4.4) equals $\delta^{(-D/2+k-1)}(\lambda)$ [41], where $\delta^{(n)}(\lambda)$ is the n th derivative of $\delta(\lambda)$. The delta-function terms in eq. (4.8) will contribute the zero-point energy; that is, such contributions will play an essentially important role in the calculation of the Casimir effect.

Consequently, eq. (9.2) should be

$$\rho(\lambda) = (4\pi)^{-D/2} \sum_{k=0, \frac{1}{2}, 1, \dots}^{\infty} \frac{B_k}{\Gamma(D/2 - k)} \lambda^{D/2-k-1} + \sum_{l=0, 1, 2, \dots}^{\infty} (4\pi)^{-D/2} B_{1+D/2+l} \delta^{(l+1)}(\lambda).$$

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